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THE USE OF IMAGINARY QUANTITIES IN ANALYTICAL GEOMETRY.

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In works on Analytical Geometry we often find the statement that for certain values of the argument x the function y beeomes imaginary and hence cannot be constructed. Yet at the same time we find that the function, although imaginary, still varies, still remains a function of x. For example take the central equation of the ellipse:

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Here y becomes imaginary for $x > a$, and yet y continues to vary as x continues to increase. The question therefore arises whether our notation is not deficient. If our notation were well adapted to represent the relations between the argument and its function, then we should be able to construct that function for *all* values of x.

By Gauss' interpretation of the imaginary numbers we know that when the "real" numbers are conceived as situate upon a straight line, then the "imaginary" numbers *must* be taken as situate upon a line right-angular to the former and having the point zero in common with it. Hence for the purposes of the Analytical Geometry of the Plane, it may be well to introduce imaginary numbers.

Let us see what notation would result from the introduction of these numbers and what advantages may be derived from it. An equation between x and y, such as equation (1), states what shall be the relative *lengths* of the abscissa and the ordinate of any point of the curve; but nothing is expressed as to the relative *directions* in which x and y shall

be counted. Therefore if, according to usage, we tacitly add the condition that y shall be perpendicular upon x, we need not be surprised if occasionally we find a construction impossible. If we wish to have the ordinates perpendicular to the abscissas, we should express this in some way. This can be conveniently done by the well known formula

$$z = x + i.y \dots\dots\dots\dots\dots\dots\dots (2),$$

z denoting the position of any point of the curve, x its abscissa, y its ordinate and $i = \sqrt{-1}$, which symbol expresses that y is perpendicular upon x. Now eliminating y from (1) and (2) we have

$$z = x + i\frac{b}{a}\sqrt{a^2 - x^2} \dots\dots\dots\dots\dots (3).$$

Here z is tho position of any point of the curve, x the abscissa and $\frac{b}{a}\sqrt{a^2 - x^2}$ the ordinate standing perpendicular upon the x axis. The curve so generated is an ellipse, as is well known.

Now let us see what becomes of the movable point z when x becomes greater than a. Put $x = a + t$, then

$$z = a + t + i\frac{b}{a}\sqrt{-(2at + t^2)} = a + t + i^2\frac{b}{a}\sqrt{2at + t^2}.$$

But $i^2 = -1$, hence $z = a + t - \frac{b}{a}\sqrt{2at + t^2} \dots\dots\dots\dots (4).$

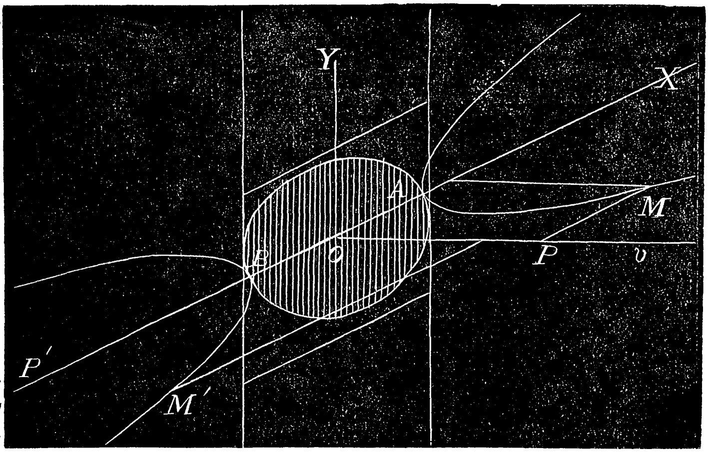
Now z, which was before of the form $\zeta + i\eta$, has become a "real" number, that is to say, z has entered the x — axis and is still a function of x as shown by equation (4), siuce $x = a + t$. The point z moves on the x axis. More strictly there are two points z, as in fact there were before x became greater than a. The ambiguity of $\sqrt{2at + t^2}$ shows that two such points z exist; and their rate of motion as depending upon t is given by equation (4).

But since all numbers, real and imaginary, occupy only two dimensions, there is in space one dimension at our disposal and nothing prevents us from constructing equation (4) in a plane at right angles to the $x\,y$ plane.

This equation (4), as we readily see, represents an hyperbola when $a + t$, as before, is the abscissa and $\frac{b}{a}\sqrt{2at + t^2}$ the ordinate. Now

we can construct equation (1) or its equivalent by our present notation, namely equation (3)

$$z = x \pm i\,\frac{b}{a}\sqrt{a^2 - x^2}.$$



This equation gives an ellipse for $x<a$ and an hyperbola for $x>a$. The plane in which the hyperbola lies is perpendicular upon the plane in which the ellipse is situated. Let xoy be a perpendicular plane, xov a horizontal plane, then the ellipse lies in the vertical plane and the corresponding hyperbola in the horizontal plane.

The equation of the hyperbola

$$y = \frac{b}{a}\sqrt{x^2 - a^2}$$

gives an ellipse for the values of x lying between $+a$ and $-a$; and an hyperbola for values of x absolutely greater than a. For, again,

$$z = x \pm i\,\frac{b}{a}\sqrt{x^2 - a^2}; \qquad \text{put } x = a - t$$

and we have, similarly to equation (4),

$$z = a - t \pm i\,\frac{b}{a}\sqrt{-2at + t^2} = a - t \pm i^2\,\frac{b}{a}\sqrt{2at - t^2}$$

$$z = a - t \mp \frac{b}{a}\sqrt{2at - t^2},$$

which is the well known vertex-equation of the ellipse, as is evident when recollecting that $a - t$ is the abscissa and $\frac{b}{a}\sqrt{2at - t^2}$ the ordinate.

The *same* figure therefore serves to show the geometrical meaning of the two equations $y = \frac{b}{a}\sqrt{a^2 - x^2}$, and $y = \frac{b}{a}\sqrt{x^2 - a^2}$.

In the same manner we find that the equation of the circle

$$y = \sqrt{a^2 - x^2}$$

represents an hyperbola for $x > a$; and that the equation of the parabola $y = \sqrt{ax}$ represents another parabola situated in a plane perpendicular upon the xy plane.

From what has been said it is evident that our notation will enable us at all times to construct the function when the ordinates are imaginary. In fact, the geometrical sense of an imaginary number being that it is to be counted in a direction perpendicular upon the line of real numbers – we see that the symbol i may be taken as nothing but a *coefficient of direction.*

By the ordinary method the equation $x^2 + y^2 + a^2 = 0$ cannot be constructed; for $y = \sqrt{-(a^2 + x^2)}$,

which shows that y is imaginary for all positive and negative values of x. But it is evident that y is a continuous function of x, and that therefore we ought to be able to construct corresponding values of x and y into some curve. Again employing our notation,

$$z = x \pm i\sqrt{-(a^2 + x^2)} = x \mp \sqrt{a^2 + x^2},$$

which gives an hyperbola.

After these introductory remarks I proceed to some more general considerations. Any curve in the xy plane may be represented by the symbol

$$z = \zeta + i\eta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

ζ being a function of x; and η another function of x. This method is preferable to putting $z = x + iy$,

where x and y are the coordinates, since by the former method functions can often be written in a more tractable form, than by the latter.

Since ζ is the abscissa and η the length of the ordinate of any point of the curve, therefore the tangent at any point of the curve will form

an angle θ with the ζ-axis whose trigonometrical tangent will be $\frac{d\eta}{d\zeta}$, *i e.*,

$$\tan\theta = \frac{d\eta}{d\zeta}.$$

We also see that the well known expressions for the lengths of the normal N, the subnormal S , the tangent T, and the subtangent S_t will remain unchanged in form, hence

$$N = \eta\sqrt{1 + \left(\frac{d\eta}{d\zeta}\right)^2}; \qquad S_n = \eta\,\frac{d\eta}{d\zeta}.$$

$$T = \eta\sqrt{1 + \left(\frac{d\zeta}{d\eta}\right)^2}; \qquad S_t = \eta.\,\frac{d\zeta}{d\eta}.$$

In short, all the relations between the differential coefficients and the form of the curve are immediately applicable to our present notation.

We will now discuss the geometrical meaning of some analytical relations known to exist between real and imaginary functions. A function of the form $f(x + iy)$ we will call (as has become customary) a *complex function;* a number of the form $x + iy$ a *complexe number.*

I. From the theory of complex functions we know that

$$\sin(x + iy) = \sin x.\cos(iy) + \cos x.\sin(iy).$$

Or, introducing for $\cos(iy)$ and $\sin(iy)$ their exponential equivalents,

$$\sin(x + iy) = \sin x.\left(\frac{e^y + e^{-y}}{2}\right) + i.\cos x\left(\frac{e^y - e^{-y}}{2}\right) \;\ldots\ldots\ldots\; (6).$$

Now x and y are independent of each other unless we establish some functional relation between these two variables. Hence this equation expresses a very general relation and will admit of the construction of a very great number of curves. Let us then assign some special value to y. At first let y be a constant number $y = a$, then (6) becomes

$$\sin(x + ia) = \sin x.\left(\frac{e^a + e^{-a}}{2}\right) + i.\cos x.\left(\frac{e^a - e^{-a}}{2}\right) \;\ldots\ldots\ldots\; (7),$$

which for the sake of brevity, we may write thus

$$\sin(x + ia) = A.\sin x + i.B.\cos x,$$

so that, comparing this with the general formula $z = \zeta + i\eta$, we see that

$$\zeta = \mathrm{A.sin}x; \qquad \eta = \mathrm{B.cos}x.$$

Now in order to find what curve is represented by (7) let ur express η as a function of ζ. Since

$$\zeta^2 = \mathrm{A}^2.\sin^2 x; \qquad \eta^2 = \mathrm{B}^2.\cos^2 x, \text{ we find}$$

$$\eta^2 = \mathrm{B}^2 \left(1 - \frac{\zeta^2}{\mathrm{A}^2}\right), \text{ hence}$$

$$\mathrm{A}^2\eta^2 + \mathrm{B}^2\zeta^2 = \mathrm{A}^2\mathrm{B}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

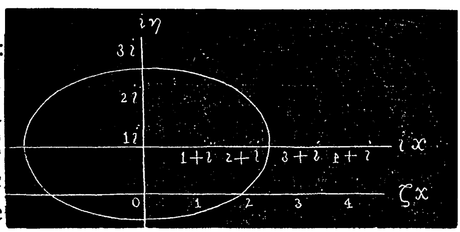
This we recognize at once as the central equation of the ellipse. No. (7) therefore is the equation of an ellipse whose major axis A $= \frac{e^a + e^{-a}}{2}$ and whose minor axis B $= \frac{e^a - e^{-a}}{2}$. Let $a = \infty$ then B $=$ A, and the curve becomes a *circle*. For $a = 0$ the imaginary part vanishes and therefore the ellipse degenerates into a *straight line*.

These results may be expressed in the following

Theorem: The equation $\qquad z = \sin(x + ia)$

represents an ellipse whose major axis is $\frac{e^a + e^{\ a}}{2}$ and whose minor axis is $\frac{e^a - e^{-a}}{2}$.

The *construction* will be this: Since the argument is $x + ia$, the imaginary part of it is constant and x describes a straight line parallel to the x-axis and at a distance from it equal to unity, namely the line in the figure marked $1\ i, i + x$. When x is 1, 2, 3, &c., then the argument is $1 + i$, $2 + i$, $3 + i$, &c. At the same time that the argument describes this straight line, the movable point z (the function) describes the ellipse. The values of the function belonging to any given value of the argument are calculated after reducing the function to the form $\zeta + i\eta$. In the present case



$$z = \tfrac{1}{2}(e^a + e^{-a}).\sin x + i\tfrac{1}{2}(e^a - e^{-a}).\cos x.$$

II. In No. (6) let x be constant, $x = a$, while y varies, then

$$\sin(a + iy) = \sin a\left(\frac{e^y + e^{-y}}{2}\right) + i.\cos a\left(\frac{e^y - e^{-y}}{2}\right)$$

Now in $z = \zeta + \iota\eta$ we have $\zeta = \sin a\left(\frac{e^y + e^{-y}}{2}\right)$ and

$\eta = \cos a\left(\frac{e^y - e^{-y}}{2}\right)$; hence to express η as a function of ζ,

$$\zeta^2 = \frac{\sin^2 a}{4}(e^{2y} + 2 + e^{-2y}),$$

$$\eta^2 = \frac{\cos^2 a}{4}(e^{2y} - 2 + e^{-2y}),$$

$$\eta^2 = \frac{\cos^2 a}{2}\left(\frac{4\zeta^2}{\sin^2 a} - 4\right),$$

$$\sin^2 a.\eta^2 = \cos^2 a.\zeta^2 - \sin^2 a \cos^2 a \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

This is the well known central equation of the *hyperbola*. For $a = 0$ we obtain a straight line parallel to the axis of ordinates whose distance from it is $\frac{1}{2}(e - \frac{1}{e})$.

For $a = \pi \div 2$ the curve becomes a similar straight line parallel to the x-axis.

Construction: The argument moves on a line which is parallel to the η-axis and at a distance from it equal to unity. This straight line passes through the intersection of the axes of the hyperbola, which point is in the x-axis at the distance $= 1$ from the origin of the coordinates. The abscissas of the hyperbola are real, the ordinates imaginary.

We now have this *theorem:*

The formula $z = \sin(a + iy)$ is the equation of an hyperbola whose axes are $\sin a$ and $\cos a$.

III. In No. (6) let $y = x$, then

$$\sin(x + ix) = \sin x\left(\frac{e^x + e^{-x}}{2}\right) + i.\cos x\left(\frac{e^x - e^{-x}}{2}\right) \ldots\ldots\ldots (10).$$

Now the argument $x + ix$ describes a straight line forming an angle of $45°$ with the x-axis, while the function describes a curve in the plane of these two lines. For $x = \pi, 2\pi, 3\pi$, &c., the abscissa vanishes, because then $\sin x = 0$, while the factor $\frac{e^x + e^{-x}}{2}$ increases very rapidly. For $x = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi$, &c., the ordinate vanishes while the fator $\frac{e^x - e^{-x}}{2}$ in-

creases very rapidly. The curve therefore is a kind of spiral. Its origine is in the origine of coordinates.

Differentiating No. (10) we have

$$\frac{d\eta}{d\zeta} = \frac{\cos x(e^x + e^{-x}) - \sin x(e^x - e^{-x})}{\sin x(e^x - e^{-x}) + \cos x(e^x + e^{-x})}.$$

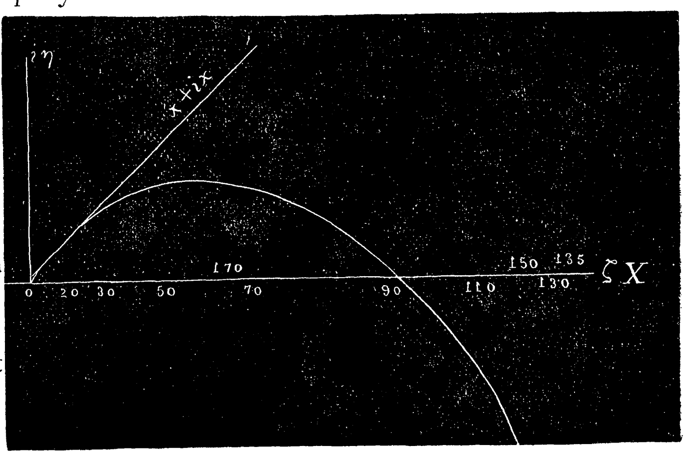
When $\sin x = 0$, then the abscissa $= 0$; but for $\sin x = 0$, $d\eta \div d\zeta = 1$, therefore the curve at every intersection with the η-axis forms an angle of $45°$ with that axis. Again, for $\cos x = 0$ the ordinate $= 0$; but for $\cos x = 0$, $d\eta \div d\zeta = -1$; therefore the curve at every intersection with the ζ-axis forms with that axis an angle of $45°$.

In order that $d\eta \div d\zeta$ may be $= 0$, we must have

$$\cos x(e^x + e^{-x}) - \sin x(e^x - e^{-x}) = 0;$$

hence
$$\tan x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

The expression on the right hand side of this equation shows that as x increases, the value of $\tan x$ approaches rapidly to unity. Hence $d\eta \div d\zeta$ will become $= 0$ for $\tan x = 1$, that is to say for $x = \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi$, &c. For $x = \frac{1}{4}\pi$ the approximation begins, but for $\frac{3}{4}\pi, \frac{5}{4}\pi$, &c., it increases very rapidly.



The construction is given in the figure. But the curve soon assumes such enormous proportions that it has been traced only a little beyond $x = 135°$ The argument describes the straight line o $x + ix$. On the axis of abscissas there are marked the points corresponding to $x = 20, 30, 50$, &c., degress.

We have thus far considered only one complex function, namely $\sin(x + iy)$, and we see that it will produce a very great number of

curves, according to the relation which we choose to establish between x and y. The same is true of all other complex functions.

To recapitulate: The introduction of the imaginary numbers into Analytical Geometry enables us to construct *all* functions of the abscissa, whether real or imaginary. For example the equations

$$y^2 + x^2 + 4y - 2x + 8 = 0,$$
$$y^4 + 3y^2 + x^2 - 5x + 10 = 0, \text{ \&c., \&c.,}$$

without the introduction of imaginary numbers cannot be constructed; but when *expressing* the condition that y is to be perpendicular upon x, it is easy to construct them. And indeed they must each represent some curve since y varies when x does.

THE PLANE TRIANGLE AND ITS SIX CIRCLES.

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The six circles whose properties are discussed in this article are *the circumscribed*, *the inscribed*, *the nine-point*, and *the three escribed circles*. The first two of these circles are familiar to every student of elementary geometry. The *nine-point circle* in a triangle is that circle whose circumference passes through the feet of the three perpendiculars from the angles upon the opposite sides, the three middle points of the sides, and the three middle points of the segments of the perpendiculars between the angles and their common point of meeting. *The escribed circles* are three circles situated wholly without the triangle, each of which is tangent to one side of the triangle and to the other two sides produced.

Three points beirg in general sufficient to determine a circumference, it is necessary to show that the nine points enumerated in the definition of the nine-point circle are always on the same circumference. To this end let ABC (Fig. 1) be a triangle, O' the centre of its circumscribed circle, $O'\alpha$, $O'\beta$, $O'\gamma$ the perpendiculars from O' to the sides BC, AC, AB, respectively. Produce $O'\alpha$ to A', $O'\beta$ to B', $O'\gamma$ to C', making $\alpha A' = O'\alpha$, $\beta B' = O'\beta$, $\gamma C' = O'\gamma$; complete the triangle $A'B'C'$, let L be the centre of its circumscribed circle, and let α', β', γ' be the intersections of AL, BL, CL with $B'C'$, $A'C'$, $A'B'$.